Bosonic Casimir effect in external magnetic field

1999 J. Phys. A: Math. Gen. 32 4457

(http://iopscience.iop.org/0305-4470/32/24/310)

Bosonic Casimir effect in external magnetic field

M V Cougo-Pinto, C Farina, M R Negrão and A C Tort

Instituto de Física, Universidade Federal do Rio de Janeiro, CP 68528, Rio de Janeiro, RJ 21945-970, Brazil

E-mail: marcus@if.ufrj.br and farina@if.ufrj.br and guida@if.ufrj.br and tort@if.ufrj.br

Received 22 December 1998

Abstract. We compute the influence of an external magnetic field on the Casimir energy of a massive charged scalar field confined between two parallel infinite plates. For this case the obtained result shows that the magnetic field inhibits the Casimir effect.

The Casimir effect can be generally defined as the effect of a non-trivial space topology on the vacuum fluctuations of relativistic quantum fields [1–4]. The corresponding change in the vacuum fluctuations appears as a shift in the vacuum energy and an associated vacuum pressure. This shift is known as the Casimir energy of the field due to the given space constraints. The original Casimir effect [5] is the attraction of two neutral perfectly conducting parallel plates placed in vacuum. The boundary conditions imposed by the metallic plates confine the vacuum fluctuations of the quantum electromagnetic field in the space between the plates. The effect of the boundary conditions can be viewed as a departure from the trivial topology of $\mathbb{R}^3$ to the topology of $\mathbb{R}^2 \times [0, a]$, where a is the distance between the plates. The resulting shift in the vacuum energy of the quantum electromagnetic field was computed by Casimir and is given by [5]:

$$\mathcal{E}_\gamma(a) = -\ell^2 \frac{\pi^2}{720a^3} \tag{1}$$

where ℓ^2 is the area of each plate and the close spacing between them is implemented by the condition $a \ll \ell$. The pressure corresponding to (1) was first measured by Sparnaay in 1958 [6] and more recently with high accuracy by Lamoreaux [7] and by Mohideen and Roy [8].

The Casimir energy has become an important ingredient of any theory with nontrivial vacuum and has been computed for fields other than the electromagnetic one with several types of boundary condition [1–4]. In the case of an electrically charged quantum field it poses by itself the question of how the charged fluctuations, and therefore the Casimir effect, are affected by fields coupling to the fluctuations through this charge. This question is strongly motivated by the fact that in a more complete picture of the Casimir effect the charged fluctuations of the constrained vacuum are, or may be put, under the influence of other fields. Within a hadron, for example, the vacuum fluctuations of quark fields are affected by the electromagnetic field of the quarks and by the colour field of gluons and quarks. Also, the vacuum fluctuations of gluon fields are affected by the colour field of quarks and gluons. In this example of quarks and gluons the unavoidable influence of the fields on the constrained charged fluctuations is of a

0305-4470/99/244457+06$19.50

extreme complexity. A reasonable realistic model for confined quarks and gluons would require fermions and bosons to be confined by boundary conditions on a sphere. The first complete calculation of the Casimir energy for the spherical geometry has been recently performed for a massive scalar field [9] and has also been extended to a massive fermionic field [10]. However, the problem for spherical geometry and an external field is certainly one of incredible difficulty. As a first step to the understanding of how charged fermionic and bosonic constrained vacuum fluctuations are affected by fields coupling to this charge we consider the problem in its most simple form, to wit: vacuum fluctuations of Dirac or scalar electrically charged fields under the influence of an external constant uniform magnetic field and constrained by the simplest of the possible boundary conditions. In the case of a constrained Dirac vacuum the external field enhances the Casimir energy [11]. Here we consider a complex scalar field confined between two infinite plates with a constant uniform magnetic field in a direction perpendicular to the plates. The charged scalar field allows us to ignore kinematical complexities which are not relevant to an initial approach to this problem. The choice of a pure magnetic field excludes the possibility of pair creation for any field strength. The confinement between infinite plates is described by a simple form of Dirichlet boundary condition and the direction of the magnetic field perpendicular to the plates is obviously a simplifying choice. Under such assumptions the formalism may be kept simple in order for us to concentrate on the fundamental issue, which is the physical effect of the external field on the Casimir effect. Once the main feature of such influence is determined the path is open to consider more complicated geometries and external fields as well as other quantum vacua. Note that we consider a problem in which the charged quantum vacuum is constrained by the boundary conditions and the external electromagnetic field is not. Therefore the influence of the external field on the charged vacuum already appears at the one-loop level, at which our calculations will be performed. In contrast, we have in the Scharnhorst effect [12–16] that boundary conditions on a pair of parallel plates are imposed on the electromagnetic quantum vacuum but not on the charged vacuum of electrons and positrons. As a result there is a change in the velocity of propagation of an external electromagnetic wave in the region between the plates. The Scharnhorst effect involves two-loop diagrams because the coupling between the external field and the quantum electromagnetic field requires the intermediation of a charged fermion loop.

Let us calculate the Casimir energy of the charged scalar field in a constant applied magnetic field using a method introduced by Schwinger to obtain the Casimir energy [17] from the proper-time representation of the effective action [18]. Since the method has been clearly explained by Schwinger [17] and already applied to several situations [19] we may use it here without going into too much detail. We start with Schwinger's proper-time formula for the effective action [18]:

$$\mathcal{W} = -\frac{\mathrm{i}}{2}\int_{s_0}^{\infty} \frac{\mathrm{d}s}{s}\, \mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}sH} \tag{2}$$

where s_0 is a cut-off in the proper-time s, Tr means the total trace and H is the proper-time Hamiltonian, which is given by $(p - eA)^2 + m^2$, where $p_\mu = -\mathrm{i}\partial_\mu$, e is the charge of the scalar field, A is the electromagnetic potential and m is the mass of the scalar field. The boundary condition gives for the component of the momentum which is perpendicular to the plates the eigenvalues $n\pi/a$,where n is a positive integer. The other spatial components of the momentum are constrained into the Landau levels created by the magnetic field $\boldsymbol{B}$ and we choose the direction of $\boldsymbol{B}$ in such a way that eB is positive. The trace in (2) is given by:

$$\mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}sH} = 2\mathrm{e}^{-\mathrm{i}sm^2} \sum_{n'=0}^{\infty} \frac{\ell^2 eB}{2\pi} \mathrm{e}^{-\mathrm{i}seB(2n'+1)} \sum_{n=1}^{\infty} \mathrm{e}^{-\mathrm{i}s(n\pi/a)^2} \int \frac{\mathrm{d}t\, \mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}s\omega^2} \tag{3}$$

where the factor two is due to the two degrees of freedom in the complex field; the first sum is over the Landau levels with the corresponding multiplicity factor due to degeneracy; the second sum is over the eigenvalues stemming from the Dirichlet boundary conditions and the integral range is given by the measurement time T and by the continuum of eigenvalues ω of the operator p^0. Following Schwinger's regularization prescription [17] we apply the Poisson sum formula [20] to the second sum in order to obtain:

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\mathrm{i}s(n\pi/a)^2} = \frac{a}{\sqrt{\mathrm{i}\pi s}} \sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}(an)^2/s} + \frac{a}{2\sqrt{\mathrm{i}\pi s}} - \frac{1}{2}. \tag{4}$$

The sum over the Landau levels is straightforward and leads to:

$$\sum_{n'=0}^{\infty} \frac{eB\ell^2}{2\pi} \mathrm{e}^{-\mathrm{i}seB(2n'+1)} = \frac{eB\ell^2}{4\pi} \operatorname{cosech}(\mathrm{i}seB). \tag{5}$$

Using (4) and (5) into (3), we obtain for the trace:

$$\operatorname{Tr} \mathrm{e}^{-\mathrm{i}sH} = \frac{a\ell^2 T}{4\pi^2} \frac{\mathrm{e}^{-\mathrm{i}sm^2}}{is^2} [1 + \mathrm{i}seB\mathcal{M}(\mathrm{i}seB)] \left[\frac{1}{2} + \frac{\sqrt{\mathrm{i}\pi s}}{2a} + \sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}(an)^2/s} \right] \tag{6}$$

where $\mathcal{M}$ is the function defined by:

$$\mathcal{M}(\xi) = \operatorname{cosech} \xi - \xi^{-1}. \tag{7}$$

Substituting now equation (6) into equation (2) we get the effective action:

$$\mathcal{W} = -2\sigma(B)\ell^2 T + \mathcal{L}^{(1)}(B) T a \ell^2 - \mathcal{E}(a, B) T \tag{8}$$

where on the right-hand side the first term is totally independent on a and is of no concern to us here, the second term gives the (unrenormalized) effective Lagrangian:

$$\mathcal{L}^{(1)}(B) = -\frac{1}{16\pi^2} \int_{s_0}^{\infty} \frac{\mathrm{d}s}{s^3} \mathrm{e}^{-\mathrm{i}sm^2} (\mathrm{i}seB) \operatorname{cosech}(\mathrm{i}seB) \tag{9}$$

and the third term gives the (still cut-off-dependent) Casimir energy:

$$\mathcal{E}(a, B) = \frac{a\ell^2}{8\pi^2} \sum_{n=1}^{\infty} \int_{s_0}^{\infty} \frac{\mathrm{d}s}{s^3} \mathrm{e}^{-\mathrm{i}sm^2 + \mathrm{i}(an)^2/s} [1 + \mathrm{i}seB\mathcal{M}(\mathrm{i}seB)] \tag{10}$$

which is the quantity we are interested in. The effective Lagrangian $\mathcal{L}^{(1)}(B)$ is analogous to the Euler–Heisenberg Lagrangian for the fermionic case [21] and was first obtained by Schwinger in 1951 [18]. Since it does not depend on a it makes no contribution to the Casimir energy. Usually, spurious terms must be subtracted before eliminating the cut-off s_o in (10) but in the present calculation they were all left in the terms of (8) which do not contribute to the Casimir energy. So we may simply take $s_o = 0$ in (10). Continuing with Schwinger's method we now use Cauchy theorem to make a $\pi/2$ clockwise rotation of the integration s-axis, which results in a substitution of s by $-\mathrm{i}s$ in the integrand of (10). Part of this integrand can be expressed in terms of the modified Bessel function K_2 (cf formula 3471.9 in [22]) and (10) reduces to:

$$\frac{\mathcal{E}(a, B)}{\ell^2} = -\frac{(am)^2}{4\pi^2 a^3} \sum_{n \in N} \frac{1}{n^2} K_2(2amn) - \frac{1}{8\pi^2 a^3} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\mathrm{d}s}{s^3} \mathrm{e}^{-s(am)^2 - n^2/s} seBa^2 \mathcal{M}(seBa^2). \tag{11}$$

The first term on the right-hand side of this equation is the usual Casimir energy in the absence of the external magnetic field:

$$\frac{\mathcal{E}(a, 0)}{a\ell^2} = -\frac{(am)^2}{4\pi^2 a^4} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2amn) \tag{12}$$

which is a result already known in current literature [1–4]; in the limit $m \to 0$ this result reduces to (1) (because the complex scalar field has two degrees of freedon and the photon field two polarizations). Here we are interested in the second term on the right-hand side of equation (11):

$$\frac{\Delta\mathcal{E}(a,B)}{\ell^2} = -\frac{1}{8\pi^2 a^3}\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\mathrm{d}s}{s^3}\mathrm{e}^{-s(am)^2-n^2/s}seBa^2\mathcal{M}(seBa^2) \tag{13}$$

which measures the influence of the external magnetic field in the Casimir energy. Due to the simple behaviour of the function $\mathcal{M}$ defined in (7) we can determine the main features of this influence. The function $-\xi\mathcal{M}(\xi)$ increases monotonically from 0 to the asymptotic value 1 when ξ goes from 0 to ∞. Therefore we see that the external magnetic field always inhibits the Casimir energy of the scalar field and suppresses it completely in the limit $B \to \infty$. This is the result that answers the question raised above. This result should be contrasted with the result for a Dirac field for which the Casimir energy is always enhanced by the external magnetic field [11]. It is very interesting that fermionic and bosonic charged vacua present such clear and opposite behaviours in the presence of an external magnetic field. We have found no intuitive explanation for this, but it is quite possibly related to the paramagnetic and diamagnetic characters of fermionic and bosonic vacua, respectively. Whatever the reason, it is important to consider this opposite behaviour of bosonic and fermionic vacua in the presence of an external field, because these vacua actually exist together in the presence of fields and may also be constrained by boundary conditions, as remarked above. Note, for example, that the shift in the zero-point energy caused by the external field depends on the mass of the field in the bosonic case (13) as well as in the fermionic case [11], and that therefore a cancellation of zero-point energies of those vacua depends on the specific relations between the masses of the quantum fields.

It is also instructive to define

$$m_B = \sqrt{m^2 + eB} \tag{14}$$

and write the complete Casimir energy (11) as:

$$\frac{\mathcal{E}(a,B)}{\ell^2} = -\frac{1}{8\pi^2 a^3}\sum_{n=1}^{\infty}\int_0^{\infty}\mathrm{d}s\, s^{-3}\mathrm{e}^{-s(am_B)^2-n^2/s}\frac{2seBa^2}{1-\mathrm{e}^{-2seBa^2}}. \tag{15}$$

Comparing this expression with its limit when $B \to 0$ we may say that the effect of the external magnetic field on the usual Casimir energy is given in the integrand of (15) by the B-dependent fraction and the constant m_B which appears in the exponential. When $B \to 0$ the fraction tends to 1 and $m_B \to m$. For a strong magnetic field the exponential is the dominant factor in the integrand and the effect of the magnetic field on the Casimir energy appears roughly as the substitution of m by m_B; certainly the B-dependent fraction may still in this case affect the precise influence of the magnetic field on the Casimir energy.

Let us consider the strong field regime, in which changes in the charged vacuum should be more prominent. The integral in equation (11) is dominated by the exponential function whose maximum is e^{-2amn} and occurs at $\sigma = am/n$. Due to this feature we are justified in substituting the function $\mathcal{M}(\xi)$ by $2\mathrm{e}^{-\xi} - \xi^{-1}$ if $B >> (\phi_0/a^2)(a/\lambda_c)$, where ϕ_0 is the fundamental flux $1/e$ and λ_c is the Compton wavelength $1/m$. Therefore, in the strong field regime, the second term in (11) can also be expressed in terms of a modified Bessel function (formula 3471.9 in [22]), and the Casimir energy can be written as:

$$\frac{\mathcal{E}(a,B)}{\ell^2} = -\frac{eBa^2}{2\pi^2 a^3}\sqrt{(am)^2+eBa^2}\sum_{n\in N}\frac{1}{n}K_1(2n\sqrt{(am)^2+eBa^2}). \tag{16}$$

Notice that the sign in the square root is to be expected because, in the regime we are working with, a minus sign means energy creation or anihilation which cannot happen when we are dealing with a constant and uniform magnetic field. We can also use (14) to rewrite (16) in the following form:

$$\frac{\mathcal{E}(a, B)}{\ell^2} = -eBa^2 \frac{am_B}{2\pi^2 a^3} \sum_{n=1}^{\infty} \frac{1}{n} K_1(2am_B n) \tag{17}$$

which is in a more appropriate form to compare with (12). By further stressing the strong field regime we can take the asymptotic limit of K_1 (cf 8446 in [22]) in (17) with $m_B \approx eB$ to obtain:

$$\frac{\mathcal{E}(a, B)}{\ell^2} = \frac{(eBa^2)^{5/4}}{a^3} \mathrm{e}^{-2\sqrt{eBa^2}}. \tag{18}$$

Turning now our attention to the weak field regime, $B \ll (\phi_0/a^2)(a/\lambda_c)$, we can substitute in the integrand of (13) $\xi\mathcal{M}(\xi)$ by $-\xi^2/6$ to obtain:

$$\frac{\Delta\mathcal{E}(a, B)}{\ell^2} = -\frac{(eBa^2)^2}{24\pi^2 a^3} \sum_{n=1}^{\infty} K_0(2amn). \tag{19}$$

Summarizing the results, we have in equation (11) the exact expression for the influence of the external magnetic field on the Casimir energy of a scalar charged field. Equations (16) and (19) particularize the result of equation (11) to the regimes of strong and weak magnetic field, respectively. In either case, the external field inhibits the Casimir energy of the scalar field. This is in contrast to the case of a Dirac field, whose Casimir energy is enhanced by the external magnetic field [11]. We may also look at the interplay between constraints and external field on the quantum vacuum from a completely different point of view. Instead of asking what is the influence of the magnetic field on the Casimir energy of the constrained vacuum we can ask what is the effect that constraints on the vacuum have on the effective Lagrangian for the magnetic field. This study has already been performed for the fermionic vacuum [23] and will in the near future be presented also for the bosonic vacuum. It would also be interesting to investigate the effect of an external magnetic field on the bosonic vacuum of a scalar field with space-time symmetry given by the κ-deformed Poincaré algebra [24] in order to see the relation between the inhibiting effect of the magnetic field on the Casimir energy and the mechanism of creation of field excitations due to the deformation.

Acknowledgments

MVC-P and CF wish to thank CNPq (The National Research Council of Brazil) and MRN thanks CAPES (Brazilian Council for Graduate Training) for partial financial support.

References

[1] Elizalde E, Odintsov S D, Romeo A, Bytsenko A A and Zerbini S 1994 *Zeta Regularization Techniques with Applications* (Singapore: World Scientific)
[2] Elizalde E 1995 *Ten Physical Applications of Spectral Zeta Function* (Berlin: Springer)
[3] Mostepanenko V M and Trunov N N 1997 *The Casimir Effect and its Applications* (Oxford: Clarendon)
Mostepanenko V M and Trunov N N 1988 *Sov. Phys. Usp.* **31** 965
[4] Plunien G, Muller B and Greiner W 1986 *Phys. Rep.* **134** 89
[5] Casimir H B G 1948 *Proc. K. Ned. Akad. Wet.* **51** 793
[6] Sparnaay M J 1958 *Physica* **24** 751
[7] Lamoreaux S K 1997 *Phys. Rev. Lett.* **78** 5
[8] Mohideen U and Roy A 1998 *Phys. Rev. Lett.* **81** 4549

[9] Bordag M, Elizalde E, Kirsten K and Leseduarte S 1997 *Phys. Rev.* D **56** 4896
[10] Elizalde E, Bordag M and Kirsten K 1998 *J. Phys. A: Math. Gen.* **31** 1743
[11] Cougo-Pinto M V, Farina C and Tort A C 1998 Fermionic Casimir effect in external magnetic field *Preprint* hep-th/9809215
[12] Scharnhorst K 1990 *Phys. Lett.* B **236** 354
[13] Barton G 1990 *Phys. Lett.* B **237** 559
[14] Barton G and Sharnhorst K 1993 *J. Phys. A: Math. Gen.* **26** 2037
[15] Cougo-Pinto M V, Farina C, Santos F C and Tort A 1999 *Phys. Lett.* B **446** 170
[16] Cougo-Pinto M V, Farina C, Santos F C and Tort A 1999 *J. Phys. A: Math. Gen.* **32** 4463
[17] Schwinger J 1992 *Lett. Math. Phys.* **24** 59
[18] Schwinger J 1951 *Phys. Rev.* **82** 664
[19] Cougo-Pinto M V, Farina C and Ségui-Santonja A J 1994 *Lett. Math. Phys.* **30** 169
Cougo-Pinto M V, Farina C and Ségui-Santonja A J 1994 *Lett. Math. Phys.* **31** 309
Cougo-Pinto M V, Farina C and Tort A 1996 *Lett. Math. Phys.* **37** 159
Cougo-Pinto M V, Farina C and Tort A 1996 *Lett. Math. Phys.* **38** 337
[20] Poisson S D 1823 *J. Ecole Polytech.* **12** 420
[21] Heisenberg W 1935 *Z. Phys.* **90** 209
Euler H and Kockel B 1935 *Naturwissensch.* **23** 246
Heisenberg W and Euler H 1936 *Z. Phys.* **98** 714
Weisskopf V S 1936 *K. Dan. Vidensk. Selsk. Mat. Fys. Medd.* **14** 3
Schwinger J 1958 *Quantum Electrodynamics* (New York: Dover) (reprint)
Miller A I 1994 *Early Quantum Electrodynamics: A Source Book* (Cambridge: University Press) (English transl.)
[22] Gradshteijn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York: Academic)
[23] Cougo-Pinto M V, Farina C, Rafelski J and Tort A 1998 *Phys. Lett.* B **434** 388
[24] Cougo-Pinto M V and Farina C 1997 *Phys. Lett.* B **391** 67